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On smoothness of solutions to structural problems and physical interpretation of weak formulation

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Subject review

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On smoothness of solutions to structural problems and physical interpretation of weak formulation

Typical structural modelling solutions are classified and described in the paper based on an approach that is more interpretative than formal. A special emphasis is placed on weak formulation and its physical interpretation which, in the authors' opinion, is lacking in the literature. The attention is also drawn to approximation errors, caused by insisting on excessive smoothness of solutions. These considerations are backed by examples. It is hoped that the paper will contribute to the understanding of the essence of approximation of practical models in civil engineering, while clearly demonstrating the power of weak formulation - the foundation of approximation procedures.

Key words:
structural model, strong formulation, weak formulation, residual, singular point, discontinuity

Pregledni rad

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O glatkoći rješenja konstruktorskih problema i fizičkoj interpretaciji slabe formulacije

U radu su na više interpretativan nego na formalan način svrstana i opisana tipična rješenja modela konstrukcije, s posebnim osvrtom na slabu formulaciju i njezino fizičko tumačenje, koje prema našemu mišljenju nedostaje u literaturi. Upozoreno je i na pogreške u približenju radi ustrajavanja na pretjeranoj glatkoći rješenja. Razmišljanja su potkrijepljena primjerima. Nadamo se da rad može pridonijeti shvaćanju biti aproksimacije praktičnih modela u građevinarstvu i zorno predočiti snagu slabe formulacije - temelja približnih postupaka proračuna.

Ključne riječi:
model konstrukcije, jaka formulacija, slaba formulacija, rezidual, singularna točka, diskontinuitet

Übersichtsarbeit

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Glatte Lösungen bei Konstruktionsproblemen und physische Interpretation schwacher Formulierungen

In der Abhandlung wurden auf eine mehr interpretative als formale Art und Weise die typischen Lösungen des Konstruktionsmodells eingeordnet und beschrieben, unter besonderer Berücksichtigung der schwachen Formulierung sowie deren physische Auslegung, die unserer Meinung nach in der Literatur fehlt. Hingewiesen wird auch auf die Fehler bei der Annäherung, um auf einer übertrieben glatten Lösung zu bestehen. Die Überlegungen werden durch Beispiele untermauert. Wir hoffen, dass die Abhandlung dem Verständnis des Kerns der Approximation der praktischen Modelle im Bauwesen beitragen und die Stärke der schwachen Formulierung – Grundlage der ungefähren Berechnungsverfahren anschaulich darlegen kann.

Schlüsselwörter:
Konstruktionsmodell, starke Formulierung, schwache Formulierung, Residuum, Singularpunkt, Diskontinuität

1. Introduction

It is well known that differential equations of linear mathematical models based on the displacement method can be symbolically written as

$$\mathcal{K}\vec{u} = \vec{f} \quad (1)$$

where $\mathcal{K}$ is a differential operator that acts on the desired displacement vector field $\vec{u}$. The result of the action is equal to a given load vector field $\vec{f}$, because the equation represents equilibrium conditions. Beside the differential equation, the solution should also meet boundary conditions for displacements. Theoretically, there is only one (exact) solution to the problem [1, 2]. Fortunately, good approximate solutions can be determined for engineering purposes [3, 4]. Moreover, there is an infinite number of them. Let's move gradually from an exact solution to a weak solution, which presents a number of advantages.

2. Strong solution

The above title indicates that such a solution, if it exists, meets a strong formulation - differential equation and boundary conditions. Results can be sought in all or in a sufficiently large number of points of the domain. The displacements are exact: in the first case they are most commonly expressed by a formula, and in the second case by a set of function values.

The set can be determined in two ways. First, it can be defined analytically, by entering the input data from the points of the domain into a closed solution, because it would be overly complicated to express it by a formula [5]. Second, it can also be determined numerically, if discrete values cannot be obtained directly, but have to be calculated (for example, iteratively, to an arbitrary precision). This is called an almost strong solution (Section 4). Through such analytically or numerically defined "cloud" of points, a polynomial or a similar function can sometimes be plotted, which can also be presented by formula. Unlike the exact one, this solution is only valid at certain points. If both formulas were valid in all points of the field, they would have to match because the exact solution of a linear model is unique. The following can be deduced from this short description: the exact solution is strong, but a strong solution can be approximate.

In some simple and "genteel" cases a strong solution can be determined directly, by solving a differential equation. Such examples, most often very idealized, are called smooth examples. They pertain to the domain of uniform properties, with no concave edges, mostly with constant boundary conditions, and to loads described by a smooth or partially smooth function. A typical example is a circular, simply supported slab of constant properties and loads. The solution, which can also be called genteel, is a fairly smooth function, as already highlighted, that is determined by a formula or a set of data that fulfils (1). In other words: the exact solution corresponds to an unbalanced load, a residual, equal to a null-vector: $\vec{r} = \vec{f} - \mathcal{K}\vec{u} = \vec{0}$. For the undeformed state $\vec{u} = \vec{0}$ and $\vec{r} = \vec{f}$, is valid, i.e. the residual is equal to the load.

Mathematically, these conditions are neither necessary nor sufficient to provide a smooth solution. However, it can be stated, purely based on experience, that fulfilment of such requirements increases the possibility of finding a solution. Since conditions for smoothness are not sufficient, they can be fulfilled, without finding a strong solution. They are also not necessary so that a strong solution can be found even without meeting such conditions. A typical example of the latter case is a model with discontinuities and points with various singularities. Together with the corresponding solution, it will be named a strong model with exceptions.

3. Strong solution with exceptions

This solution is smooth everywhere, except at some points where the values of displacements, rotations, deformations, and/or stresses tend to infinity. These are the points of the concave corners, with the concentrated forces or moments, the points of sudden change in thickness, modulus of elasticity, boundary conditions, etc. Physically, these are the areas of stress concentration. In a singular point the differential equation and/or its assumptions are not fulfilled. At such points the displacement field is often not derivable enough to be inserted into a differential equion.

An example is a simply supported slab, shaped as a three-quarter circle, with constant properties (Figure 1). Figure 1.a shows one deflection as possible solution, and Figure 1.b the corresponding load, determined by the inverse method. At the point of a concave boundary, the derivatives of the displacement field (internal forces and loads) tend to infinity. With the exception of the

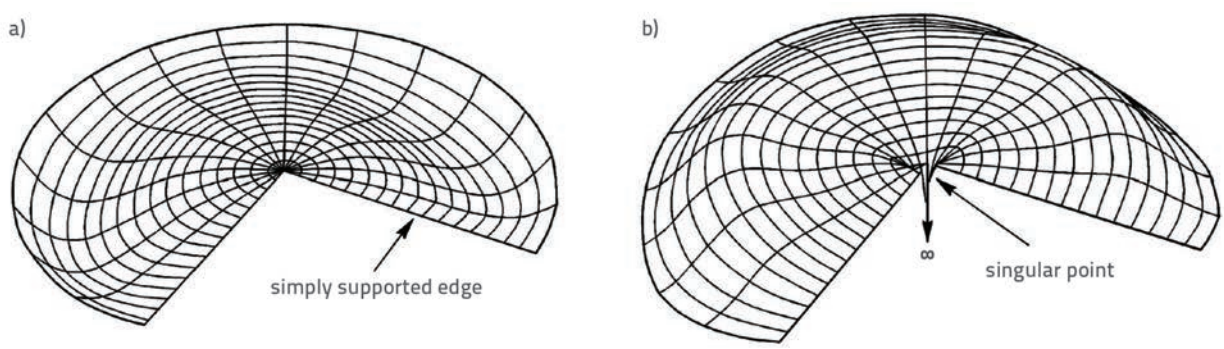

Figure 1. Simply supported slab: a) possible deflection surface, b) the corresponding load

singular point, such a solution is like a strong or almost strong solution; it satisfies the boundary problem equations.

The mathematical accuracy of the solution does not guarantee its physical correctness. Often, there are (usually small) parts of the domain where contradictions exist: the differential equation solution is correct, but the assumptions necessary for its formation are violated. For example, small displacements and deformations are assumed by geometric linearization but, at some points, excessively large or infinite values are provided in the solution. Or the validity of Hooke's law has been assumed, and computed deformations and stresses enter into the plastic part of the material model. Inconsistent regions are usually in the vicinity of singular points, but they may be farther away, even in smooth examples. Hooke's law can be violated in a completely smooth place.

Apart from the area of distribution, the influence of such disturbances is also limited. According to Saint-Venant's principle, it has a significant local, and negligible global influence on the resulting fields. Structural engineers may implement such solutions, but they must be aware of the consequences and carry out additional local checks, correct the obtained values, use more precise modelling at singular regions, round concave edges, and often apply structural measures (use of strengthening elements, stiffeners and haunches) around such points.

Many strong solutions (traditional ones and solutions with exceptions) can be generated by inverse or semi-inverse methods [6]. So determined, they usually have no practical or theoretical significance, but they are used for verification of numerical methods.

4. Almost strong solution

This solution, that can be marked as $\vec{u}_p$, approximately satisfies the differential equation and boundary conditions at each point of the domain:

$$\mathcal{K}\vec{u}_p \approx \vec{f} \tag{2}$$

If $\vec{u}_p$ is put into the strong form, two questions must be asked. First: Is the approximate solution sufficiently derivable to be inserted into the equation? And second: If so, is the equation approximately satisfied? The second question makes sense only if the answer to the first one is positive, because the derivability can not be discussed if the numerical solution is defined by a series of discrete values. If the solution can be inserted in (2), the validity can be evaluated with the residual.

$$\vec{r} = \vec{f} - \mathcal{K}\vec{u}_p \neq \vec{0} \tag{3}$$

Because of approximate nature of $\vec{u}_p$, the part of the load is not balanced. If the inverse method is applied to $\vec{u}_p$, we get $\mathcal{K}\vec{u}_p = \vec{f}_p$, which gives the corresponding load $\vec{f}_p$. Of course, it differs from $\vec{f}$ (otherwise $\vec{u}_p$ would be the exact solution). Now according to (3) the residual can be written as follows:

$$\vec{r} = \vec{f} - \vec{f}_p \neq \vec{0} \tag{4}$$

If the approximate solution is good then $\vec{r} \approx \vec{0}$ and, as with most approximations, the error must be less than the criterion, i.e. $\| \vec{r} \| < \varepsilon$. Although we have highlighted in Section 2 in the paragraph relating to smooth examples that the almost strong solution is as close as we wish it to be to the strong solution, the almost strong solution is mostly identified with a numerical solution that approximates the strong one very well.

In special cases, if the exact solution lies in the space of coordinate functions, the numerical approach does not give an almost strong solution but rather a strong solution. For example, if the exact solution is a polynomial, and the coordinate functions form a complete set of polynomials, the Ritz method [7] gives a strong solution. If the exact solution is not a polynomial, but it can properly be approximated with a polynomial, we get an almost strong solution. It can also be used for problems with exceptions, but then the space of coordinate functions must contain singularities in the places where they appear in the exact solution. Such functions are rarely chosen in practical situations. The finite difference method [8] also provides an almost strong solution in all instances, except near the singularities and discontinuities. In regular points, the exact solution can be expanded into convergent Taylor series, and the solution derivatives can in such cases be reasonably well approximated by finite differences. Other numerical methods also give an almost strong solution, if the exact solution is a sufficiently (or at least partly) smooth function.

Unfortunately, some numerical methods are incapable of finding an almost strong solution of smooth examples, because the process itself generates parasitic discontinuities (those that are not inevitable, inherent to the problem, but are the consequence of method selection). For example, the finite element mesh [9, 10] often contains discontinuities at the boundary of two elements. Such coordinate functions can then be regarded as non-smooth.

5. Excessively strong approach to the solution

A strong solution also satisfies an excessively strong formulation: not just a differential equation, but also its derivatives, or additional equations formed by single and multiple derivation of the basic equation. That means, except in the singular points, that the solution may have a higher smoothness than is necessary for a strong or almost strong solution. The displacements have more continuous derivatives than is required by the equation. This approach is sometimes used in numerical methods.

If applied to extremely smooth examples, this strategy does ensure better convergence. If the solution is a smooth, analytic function, with no singular points, it can even be infinitely differentiable. Solution to a very smooth problem is then simple and, with excessive assumptions, the accuracy of a strong solution is achieved with a smaller number of unknowns.

However, forcing an excessively strong solution can create difficulties in the case of exceptional points. The reason for this is simple: by differentiating a strong solution with exceptions, we often increase the number of discontinuities. If there is a chance for a singularity, it will surely appear through this approach (Figure 2). For example, differentiation of an equation for a plate, partially loaded with a constant load, creates new discontinuities at the boundary of the loaded and the unloaded parts.

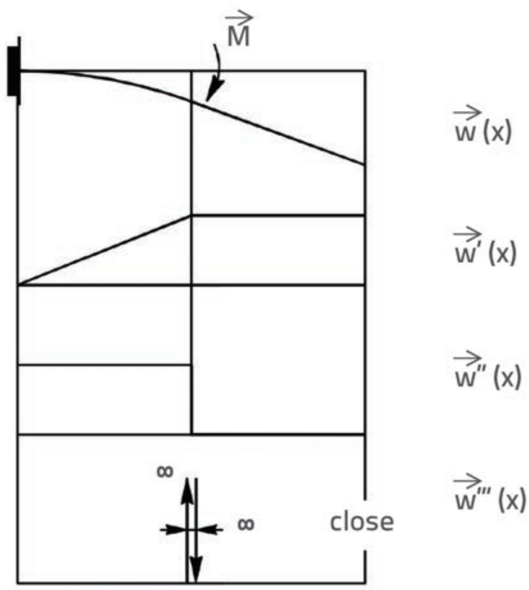

Figure 2. An example of a singular point

A very strong approach is emphasized here because it is sometimes believed that it speeds up convergence. An equally good approximation is obtained with shorter computing time (crude discretization, larger load or time steps), or a much better approximation is achieved within an equal time frame.

The truth is: if we insist on excessive smoothness, it speeds up solution of smooth problems but slows down the search for solutions with singular points and discontinuities of the region or function we try to approximate. More precisely, problems where the Taylor series is convergent are good candidates for acceleration, while acceleration is not possible where it is divergent or poorly convergent (we need a large number of terms for an acceptable solution). In such a case, by forcing smoothness, we deteriorate the solution process.

An example of an excessively strong approach. In order to advise the readers in greater detail about this aspiration toward using an excessively strong solution, it is of interest to consult the known equation on the motion of the model with one degree of freedom:

$$m\frac{d^2u}{dt^2}+c\frac{du}{dt}+ku=f\left(t\right) \tag{5}$$

where m, c and k are the mass, damping, and stiffness, respectively, u denotes displacement, the first and second derivatives are velocity and acceleration, and $f(t)$ is the load function. In the non-linear case constants may also depend, for example, on time, displacement and velocity, and so the equation can generally be written as:

$$\frac{d^2u}{dt^2}=g\left(t,u,\frac{du}{dt}\right) \tag{6}$$

where g is a function of such dependencies. If initial conditions at time $t = t_0$ are

$$u(t) = u_0 \quad \text{and} \quad \frac{du}{dt}(t_0)=\left(\frac{du}{dt}\right)_0 \tag{7}$$

we can include them in the differential equation and determine the initial value of the second derivative:

$$\left(\frac{d^2u}{dt^2}\right)_0=\frac{d^2u}{dt^2}(t_0)=g\left(t_0,u_0,\left(\frac{du}{dt}\right)_0\right) \tag{8}$$

This is the normal accuracy that is required in the realization of time discretization procedures. For instance, it can be used for direct integration of dynamic equation [11, 12]. The increased accuracy is obtained if we derive the left and right sides with respect to time, and determine initial values of the third and higher derivatives, up to an appropriate order, depending on the desired degree of approximation:

$$\left(\frac{d^3u}{dt^3}\right)_0=\frac{d^3u}{dt^3}(t_0)=\frac{d}{dt}g\left(t_0,u_0,\left(\frac{du}{dt}\right)_0\right)$$

$$\left(\frac{d^4u}{dt^4}\right)_0=\frac{d^4u}{dt^4}(t_0)=\frac{d^2}{dt^2}g\left(t_0,u_0,\left(\frac{du}{dt}\right)_0\right) \tag{9}$$

...

$$\left(\frac{d^Su}{dt^S}\right)_0=\frac{d^Su}{dt^S}(t_0)=\frac{d^{S-2}}{dt^{S-2}}g\left(t_0,u_0,\left(\frac{du}{dt}\right)_0\right)$$

Values can be included in Taylor series:

$$u(t)=u_0+\left(\frac{du}{dt}\right)_0(t-t_0)+\frac{1}{2}\left(\frac{d^2u}{dt^2}\right)_0(t-t_0)^2 + \frac{1}{3!}\left(\frac{d^3u}{dt^3}\right)_0(t-t_0)^3+\ldots+\frac{1}{s!}\left(\frac{d^su}{dt^s}\right)_0(t-t_0)^s + \mathcal{O}\left((t-t_0)^{s+1}\right) \tag{10}$$

If time is calculated from $t_0 = 0$, and the difference $t - t_0$ is labelled with Δt, and if contribution of terms of the order $\mathcal{O}(\Delta t^{s+1})$, is omitted, the development can serve as an explicit algorithm for solving the dynamic equation. The solution at the end of the first step (with a known initial displacement and derivatives) can be expressed as:

$$u(\Delta t)=u_0+\left(\frac{du}{dt}\right)_0+\Delta t+\frac{1}{2}\left(\frac{d^2u}{dt^2}\right)_0\Delta t^2 + \frac{1}{3!}\left(\frac{d^3u}{dt^3}\right)_0\Delta t^3+\ldots+\frac{1}{s!}\left(\frac{d^su}{dt^s}\right)_0\Delta t^s \tag{11}$$

At the end of the *n*-th step (with a known solution from the previous step), we have:

$$u(n\Delta t) = u_{n-1} + \left(\frac{du}{dt}\right)_{n-1} + \Delta t + \frac{1}{2}\left(\frac{d^2u}{dt^2}\right)_{n-1} \Delta t^2 + \frac{1}{3!}\left(\frac{d^3u}{dt^3}\right)_{n-1} \Delta t^3 + \ldots + \frac{1}{s!}\left(\frac{d^su}{dt^s}\right)_{n-1} \Delta t^s \quad (12)$$

In very smooth cases, the numerical solution can be improved by this approach, but with a "strong" assumption: higher derivatives of the displacement function (higher order terms of the Taylor series expansion) have to exist. Furthermore, the physical interpretation of higher derivatives is also not clear.

6. Weak solution

Unfortunately, an almost strong solution, and especially a strong solution, cannot be found for many engineering problems, and so it is logical to ask a sub-question: If the questions from Section 4 are not fulfilled, is the numerical solution necessarily bad? The answer is optimistic: it is not. The differential equation and boundary conditions do not have to be approximately satisfied at all points. It is enough that they are valid integrally, at arbitrary parts of the region. The requirement $\vec{r} \approx \vec{0}$ that corresponds to the almost strong solution is too restrictive. As to the strong solution for which $\vec{r} = \vec{0}$ is valid, it should not even be argued about. That is, it is not needed to approximate or completely satisfy the equilibrium conditions at each point of the model. It is known that structures are not sensitive to local overloads, if the values are not too large (as in punching shear problems for example). Dead and live loads on slabs are prescribed uniformly, although significantly larger values are possible, such as grouping of furniture (bookcases) or people (during a meeting). And nothing terrible happens to the slab. Here, a nice idea arises: instead of the exact one, a good approximate load can be used.

6.1. Notion of substitute load

Substitution is so natural and intuitive for engineers that they do not even think of the consequences [13]. Difficulties arise during formal interpretation on mathematical models. Here are some examples when, without discussing residuals, a substitute load is applied. In doing so, we substitute continuous load with discrete load and vice versa.

The differential equation of a beam is easier to solve if load is presented by a smooth continuous function (such as a polynomial or a harmonic function) than if it is modelled with concentrated forces and moments. In engineering standards, the load of pedestrians or vehicles is replaced by a continuous one. In graphic statics it is the opposite: for graphical constructions, the continuous load is represented by a series of concentrated forces. In numerical methods, it is very similar. The loads in the finite element method always act as forces at the joints. During laboratory tests and load testing, the load is realized with a layer of gravel, with stone material, water barrels, sandbags, vehicles, despite the fact that it is prescribed as uniformly distributed.

According to fundamental mechanics, the loads are considered similar if their resultants are approximately equal. Equivalently, difference in total loads – the residual, is approximately equal to the null-function:

$$\int_{\Omega} \left(\vec{f} - \vec{f}_p\right) d\Omega = \int_{\Omega} \vec{r} d\Omega \approx \vec{0} \quad (13)$$

This condition obviously does not also mean that $\vec{r} \approx \vec{0}$. The residual is a vector function, so for small values over the domain it is not enough to require a small resultant (integral). For example, the residual may be an antisymmetric function (Figure 3). The areas underneath the curve are cancelled out by integration; therefore, the resultant is equal to the null-function, while the residual is obviously not. The above expression, given in the form of a sum over arbitrarily selected regions, is also not sufficient:

$$\sum_{i=1}^{n} \int_{\Omega} \vec{r} d\Omega_i \approx \vec{0} \quad (14)$$

where $\cup\Omega_i = \Omega$. Using this formulation, the sum of the resultants (of residual) per region must be small, but again, some terms may be large opposite vectors that cancel each other in the sum. Obviously, additional requirement is missing: each resultant must be sufficiently small. This will also keep the sum close to the null-vector, and (13) and (14) will be satisfied

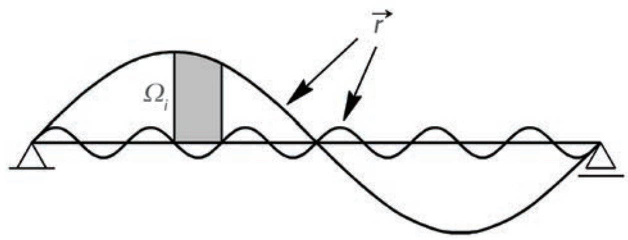

Figure 3. Good and bad residual function

There is a significant difference between the solution based on this strategy and the almost strong solution. The additional requirement means a small residual resultant over every region Ω_i, not a small residual ordinate at each point of the domain. This approach guarantees a similar distribution of the given load and substitute load, similar resultants, (only) over each subdomain. The function of the residual that allows such substitute load is added to Figure 3. This solution strategy leads directly to the weak form. It should be pointed out: the strong and the almost strong solution trivially satisfy (13) and (14), because the residual is everywhere equal to (or close to) the null-vector.

6.2. Weak form

Formal understanding of previous explanations is not difficult but, in our opinion, a connection with practical examples is missing. That is why we have decided to further reduce the level of abstraction, generality and rigidity, in an attempt to preserve mathematical correctness as much as possible. We tried to clarify the notion of a weak form by applying three simple examples. Of course, they are not a substitute for theory, but they help to illuminate issues from the other, physical side. After these considerations, the fundamental lemma of calculus of variations and the consequences of its introduction into numerical methods should be studied [14-16].

6.2.1. The experiment as motivation

Let's imagine a simple task: one should test a uniformly loaded reinforced concrete slab. How can we achieve a constant load? Water is a good realization of a uniform load (Figure 4). On a rigid, horizontal slab, the depth of water is constant, so the load is the same at all points. Thinking more precisely, that is not exactly true. The deflection of the slab causes the change of depth and the load ordinate. If the slab is too thin, slender, deflections may be large, so we would have to take into account the non-linear relationship between the displacement and load. Even with a rigid slab, the realization of loading with water (due to formation of a watertight pool) would be quite inappropriate.

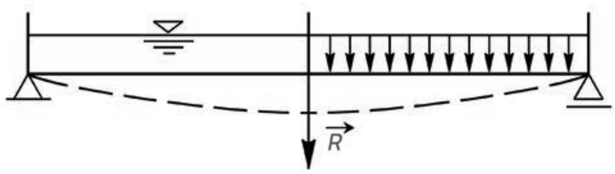

Figure 4. Water load model

Instead of water, it would be easier to realize the load with gravel (Figure 5). However, the slab needs to be enclosed again, but it doesn't need to be watertight, and the application and removal of gravel is easier to implement. The total weights of the material and the uniformly distributed load (marked with $\vec{R}$) have to match. The gravel should be uniformly distributed over the slab. With the first requirement we fulfil the integrals (13) and (14), and with the second one the additional requirement of equal loads on smaller regions of the slab. The gravel load must match the shape of the uniform distribution as well.

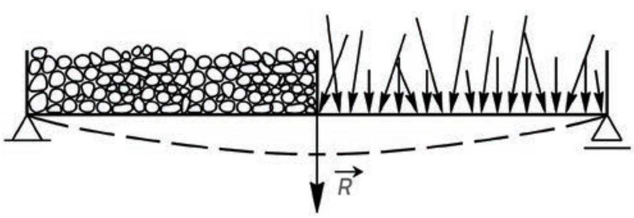

Figure 5. Gravel load model

Physically speaking, this substitution is very natural, and such an idea has been confirmed byexperience and experiments. Measured displacements and deformations (within the measurement accuracy limits) do not depend on load distribution. Regardless of whether the loading is realized by gravel or water, even very precise instruments will not register any difference.

However, the load on the slab at the grain level deviates significantly from a uniform pressure. It consists of irregularly distributed contact stresses, which highly oscillate around a uniform value. At the contact between grain and the slab, the load can be very large, and between the contacts there is no touching (or loading) at all. If the slab and grains are absolutely rigid (non-deformable), the contact is realized at the point through which the contact force passes, and the stress at that place tends to infinity. In the case of real materials, the contact surface increases. Contact stresses develop, and the contact is realized over the small surface. Because grains support each other, the stresses act at an angle, since the roughness of the slab surface causes friction. Thus, in contrast to water, the slab is also loaded by horizontal forces (in addition to the vertical ones; see Figure 5). According to the equilibrium equations, the resultant of the horizontal forces equals zero, and that of the vertical corresponds to the weight of the gravel. The same applies to any part of the slab. If the horizontal forces are reduced to the neutral plane, the moments equal to the product of horizontal forces and half the thickness of the slab act on it, but their sum is also zero. Vertical contact stresses affect distribution of normal stresses, perpendicular to the slab, but are not relevant for the design and according to the thin slab theory they are usually omitted.

A much coarser substitute load can be applied in the case of large structural dimensions as related to elements the load is applied to: we emphasized the substitution of a continuous load with concentrated forces, as well as stones, barrels, vehicles and the like. All loads differ from the uniformly distributed load considerably but, according to measurement results, they are practically equivalent. For example, with the funicular polygon we substitute a uniformly distributed load with a series of concentrated forces. At the beam points level, it is a very bad approximation. In almost all points, the vertical stress is equal to zero, and in a small number of points where the forces act, it tends to infinity. However, with an increase in the number and decrease in the spacing of concentrated forces, the approximation of the internal forces converges towards the solution for continuous load. A trivial example is shown in Figure 6.

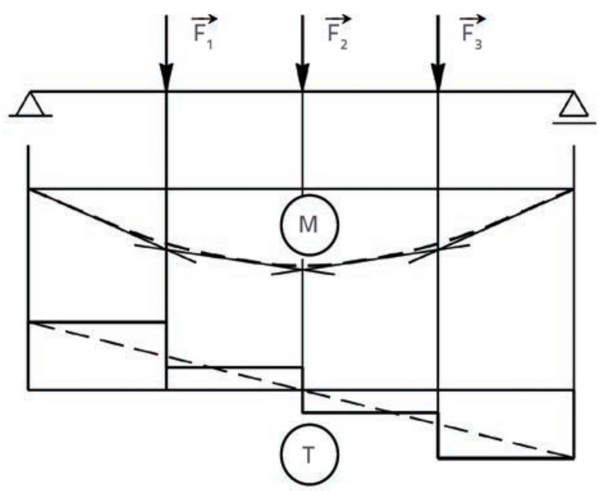

Figure 6. Substitution of a continuous load with concentrated forces

As we can see, the intuition gives different answers if we are solving an abstract mathematical problem, the differential equation of the slab, or if we are carrying out the practical realization of experiment. Although, actually, it is the same slab. When we are thinking about approximate solution, we try to fulfil it in the best possible way, and so the (mathematical) intuition requires the lowest possible difference between the substitute load and given load. By achieving such a goal, the equation is approximately satisfied at each point, i.e. we get an almost strong solution. On the contrary, if we are thinking about a real slab, the (physical) intuition considers that a discontinuous, very rough load, is completely appropriate. Physical and mathematical intuition are thus in clear contradiction. From the results of experiments and mathematical proofs, we know that the first one is good enough, while the second one is restrictive and overly rigorous.

In terms of a strong or almost strong form, the layer of grains is not a good approximation of a uniformly distributed load. The differential equation is valid only for a small number of random points where the load value of grains almost or completely cancels the uniform load. Only in such places the residual is close to or equal to zero. Notice the null-points of residual functions in Figure 3.

Luckily, the gravel is levelled, and such a function satisfies a much weaker but sufficient condition: the integral of the residual (resultant) on any piece of the slab is very small. In this respect, the region must carry a sufficiently large number of grains so that the weight of the gravel is close to the resultant of a uniformly distributed load. Then, the resultant will change negligibly by random removal or addition of a few grains. If a very small piece of the slab is considered, it is loaded by several grains or by no grains at all, so the difference to the resultant of the constant load on such a small piece is usually enormous. Then even the change in the number of grains can have a significant effect on the resultant. However, the piece must be small enough, because the local load distribution must not significantly affect the entire slab. So, the shape of the slab pieces is not important, but the relation between the size of the piece and the size of the grain is.

The influence of the nonuniform segment load on distant parts of the slab can, according to the Saint–Venant principle, be replaced by the influence of a uniformly distributed segment load. If we want to repeat the experiment with the gravel layer again, the contacts would appear at completely different points and, in the sense of the strong form, loading would be different from the old one. There are infinitely many such equivalent loads, but the deflections and stresses are practically the same. If experiments are repeated by gradually reducing the size of gravel (and by ultimately using sand and dust), the number of concentrated forces and moments increases, while the spacing between them decreases. The residual function becomes more and more jagged, but its extreme values decrease. Displacements, internal forces and other values approach the correct values.

After these considerations, we can conclude: if residual function fulfils these conditions, the corresponding solution is called weak. The word "weak" may suggest that such a solution is worse than the "strong" one. We can see that it is not. Even if the differential equation is not approximately satisfied, the exact values approximate quite well the displacements, internal forces, and stresses obtained by the weak approach. Indeed, the weak solution is obtained by integrating the function of the substitute load, and by such an operation on a "rough" function, we are "smoothing" it. The more times we integrate the function, the smoother it becomes, and the approximation is better. Thus, despite the poor approximation of loading, already the shear force, obtained by the integration of loads, approximate the exact values much better. Bending moments are better approximated than shears, angles of rotation are approximated even better, while displacements are approximated the best. Let us consider some simple examples.

6.2.2. Strength of weak form

In addition to the spatial example with gravel, we decided to demonstrate the strength of the weak form on two simple planar models, which can exactly be solved for the given and substitute loads. First let us take a look at the simply supported beam with the span $\ell = 1$, subjected to the unit continuous load $\vec{q}(x) = \vec{k}$. The substitute load is chosen in the form of

$$\vec{q}_p(x) = \left[1 + \sin(n\pi x)\right]\vec{k} \tag{15}$$

where n is the number of half waves. We can consider it as a gravel load with idealized grains. Let us define the function of the residual with the expression

$$\vec{r}(x) = \vec{q}_p(x) - \vec{q}(x) = \sin(n\pi x)\vec{k} \tag{16}$$

as if gravel is the given load, and the uniform loading is the substitute load. The extremes of the residual are ±1 and they do not depend on the number of waves. The integral of the residual load along the span is for the even n equal to the null-vector, and for the odd n is equal to the resultant of the half wave:

$$\int_0^{\ell} \vec{r}(x)dx = \int_0^{\ell} \sin(n\pi x)dx\vec{k} = \begin{cases} \vec{0} & \text{for even } n, \\ \pm 2/(n\pi)\vec{k} & \text{for odd } n. \end{cases} \tag{17}$$

The value of the integral decreases with an increase of n. In the limiting case, when $n \to \infty$, grains become smaller and smaller, and the value of the integral tends to zero. Relative errors, obtained by integrating residuals as loads, are given in Table 1.

Table 1. Error of beam values

Relative error of beam values		
Error of	**Definition of error**	**Value**
loading	$\max(\lVert\vec{r}(x)\rVert / \lVert\vec{q}(x)\rVert)$	1
shear force	$\max(\lVert\Delta\vec{T}(x)\rVert / \lVert\vec{T}(x)\rVert)$	$1/(2n\pi)$
moment	$\max(\lVert\Delta\vec{M}(x)\rVert / \lVert\vec{M}(x)\rVert)$	$1/(2n\pi)^2$
rotation	$\max(\lVert\Delta\vec{\varphi}(x)\rVert / \lVert\vec{\varphi}(x)\rVert)$	$1/(2n\pi)^3$
displacement	$\max(\lVert\Delta\vec{w}(x)\rVert / \lVert\vec{w}(x)\rVert)$	$1/(2n\pi)^4$

Numerators with the sign Δ denote the residual of the beam magnitude, i.e. the difference in values that correspond to the substitute load and given load. For example, for the shear force we have

$$\Delta\vec{T}(x)=\vec{T}_p(x)-\vec{T}(x) \tag{18}$$

We can see constant maximum values of residuals (errors in load). With an increase in n the frequency of the residual function also increases and becomes more jagged, but the extremes remain the same. Internal forces and displacements due to such loads decrease in the way that is inversely proportional to the number of half waves; their errors tend to zero with different powers (the last column of the table).

At the same time, the load function becomes discontinuous at each point, and in the limit, it tends towards a non-derivable function, because it contains infinitely many null-points, spikes and jumps. Nevertheless, deflection and forces from such load are exactly defined continuous functions, because they completely satisfy the differential equation. And not only that. In the limit, the solutions match those for constant action. Obviously, the load becomes rougher, and the results get better. The substitute load with five and twenty half waves, and the corresponding internal forces, are shown in Figures 7.a) and b).

It looks as if the beam (can also be a slab or shell) is not sensitive to an increase in jaggedness of the load. We have already noted that such behaviour is expected because the integration, used to determine internal forces, smooths the load function. Physically, the simple beam behaves under residual load like a continuous beam with spans $1/n$. The errors correspond to the distribution of beam values along such short spans. In the limit, all (spans, moments, and other values) tend to zero. If we determine the flexural and shear reinforcement by applying crude diagrams of internal forces, which correspond to the very bad "uniformly distributed continuous" load (Figure 7.a), the beam will be safe enough, and the difference in the reinforcement towards the "really" uniform action can be considered negligible, especially since the number of bars is an integer.

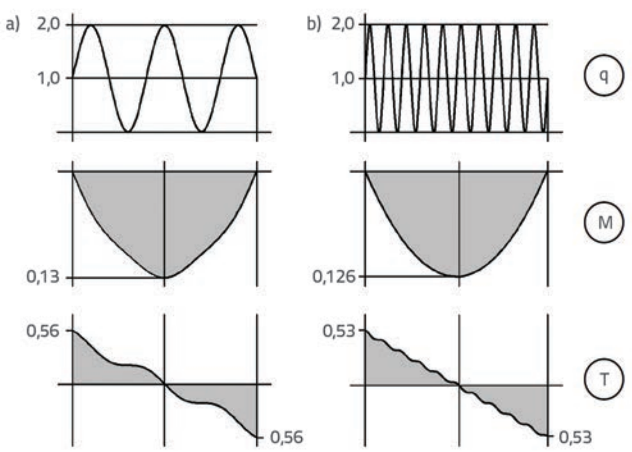

Figure 7. Internal forces for rough approximation of uniformly distributed load

Let us look once again at the simply supported beam that is now loaded with a unit force in the middle. The substitute load can be determined by the Fourier series expansion of concentrated force (Figure 8):

$$\begin{aligned}\vec{q}_p(x)&=2\left[\sin(\pi x)-\sin(3\pi x)+\sin(5\pi x)-\ldots\right]\vec{k}\\&=2\left[\sum_{i=1}^{\infty}(-1)^{i+1}\sin(2i-1)\pi x\right]\vec{k}\end{aligned} \tag{19}$$

The load is continuous, the series is divergent, and the sum tends to infinity. However, the shear forces and moments of this loading approximate the exact values quite well. Figures 8.a and 8.b show solutions if force is approximated by five and forty terms.

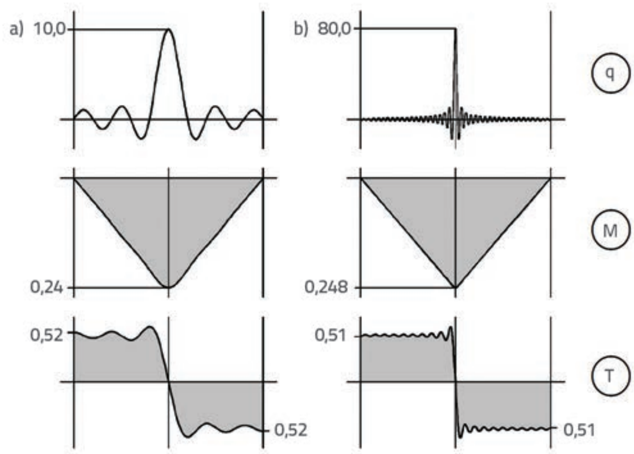

Figure 8. Internal forces for crude approximation of concentrated load

Note, the load drawing scale in this example is not the same. If scales were the same, the waves of the load function, with the exception of the central spike, would be roughly equal.

7. Final considerations

Let us also note the known mathematical expression of the weak form. If we make a dot product of the residual of strong solution (the vector null function) and any vector function $\vec{g}$, the result will be a scalar null function. And a definite integral of such a function is equal to zero. Therefore,

$$\int_{\Omega}\vec{r}\ \vec{g}\ d\Omega=0 \tag{20}$$

where $\vec{g}$ is a test function [9]. The integral represents the scalar product of vector functions, and we can interpret it similarly to a scalar product of vectors. The scalar product for the arbitrary value and the direction of one vector (in this case $\vec{g}$) is equal to zero, only if the second one (i.e. $\vec{r}$) is equal to the null-vector. This approach to the boundary problem is called a weak form, and the solution that satisfies it is considered weak.

It's easy to notice: if we know the solution of the differential equation, then $\vec{r} = \vec{0}$, and value of (20) is also zero. So, the strong solution fulfils the weak form. From the previous examples we know that the reverse does not apply. The weak solution does not fulfil the differential equation. Even if we could insert it, we would get large residuals. In terms of the inverse method, it is the exact solution for the substitute load (gravel), and it is quite different from the solution for the given uniformly distributed load (water). Since the residual is a very non-smooth function, difficulties may arise with integration in expression (20). In such cases, the Lebesgue integral is used instead of the Riemann integral [17].

In practice, a strong form is rarely satisfied. But the weak form can almost always be fulfilled for a set of chosen test functions. Let us emphasize once again: this cannot guarantee the equilibrium of each point at the body, but only the equilibrium of any piece regardless of the way in which it is cut off. The corresponding residual can always be presented by an irregular, "rough" function with breaks and jumps. Despite all this, it is a physically correct solution.

An advantage should not be given to the strong form for one more reason. There are numerous reasons why mathematical models are not precise descriptions of real engineering problems [18]. Thus, even the exact solution of the model is not a solution to a physical problem [19]. We do not know this solution and we will never know it. In a building, each floor can have a different purpose and loading: imagine a sculptor's studio, a dance school, or a martial arts course, and the slabs are always the same, designed for the same actions. Why then insist on precision? It is interesting to note: the material's core is closer to the weak form than to the strong form. This is analogous to replacing a discontinuum with a continuum and vice versa.

From all these considerations it can be concluded that it is easier to satisfy the weak form, or to find a solution from a larger set of appropriate solutions, rather than to look for the exact, often very complex and the only one strong solution. And from the mathematical point of view, the integral formulation is easier to solve than the differential but also, in the light of application of computers, scalar variables are more suitable than vector ones. It is fairly known that expression (20) allows partial integration which, in relation to differential formulation (1), reduces the order of derivatives and smoothness of the solution [15]. In the end, in the theory of structures, the weak form is equivalent to the principle of virtual work [20, 21].

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